SOME RESULTS ON ZEROS DISTRIBUTIONS AND UNIQUENESS OF DERIVATIVES OF DIFFERENCE POLYNOMIALS

KAI LIU, XINLING LIU, TINGBIN CAO

Abstract. We consider the zeros distributions on the derivatives of difference polynomials of meromorphic functions, and present some results which can be seen as the discrete analogues of Hayman conjecture [8], also partly answer the question given in [18, P448]. We also investigate the uniqueness problems of difference-differential polynomials of entire functions sharing one common value. These theorems improve the results of Luo and Lin[18] and some results of present authors [15].

1. Introduction

In this paper, a meromorphic function f means meromorphic in the complex plane. If no poles occur, then f reduces to an entire function. Throughout of this paper, we denote by $\rho(f)$ and $\rho_2(f)$ the order of f and the hyper order of f [10, 26]. In addition, if $f-a$ and $g-a$ have the same zeros, then we say that f and g share the value a IM (ignoring multiplicities). If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then f and g share the value a CM (counting multiplicities). We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory [9, 10, 26].

Given a meromorphic function $f(z)$, recall that $\alpha(z)\not\equiv 0,\infty$ is a small function with respect to $f(z)$, if $T(r,\alpha)=S(r,f)$, where $S(r,f)$ is used to denote any quantity satisfying $S(r,f)=o(T(r,f))$, and $r\to\infty$ outside of a possible exceptional set of finite logarithmic measure.

The following result is related to Hayman conjecture [8, Theorem 10] which has been considered in several papers later, such as [1, 2, 19].

Theorem A. [2, Theorem 1] Let f be a transcendental meromorphic function. If $n\geq 1$ is a positive integer, then $f^nf'-1$ has infinitely many zeros.

Remark that $[f^{n+1}]'=(n+1)f^nf'$ in Theorem A, Chen [3], Wang and Fang [22, 23] improved Theorem A by proving the following result.

Theorem B. Let f be a transcendental entire function, n,k be two positive integers with $n\geq k+1$. Then $(f^n)^{(k)}-1$ has infinitely many zeros.

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Laine and Yang [11] firstly investigated the zeros of $f(z)^n f(z+c)$ and proved the following result.

Theorem C. Let f be a transcendental entire function of finite order and c be nonzero complex constant. If $n \geq 2$, then $f(z)^n f(z+c) - a$ has infinitely many zeros, where $a \in \mathbb{C}\backslash\{0\}$.

Recently, some papers are devoting to improve Theorem C, the constant a can be replaced by a nonzero polynomial [12] or by a small function $a(z)$ [15]. In addition, [13, 14, 18, 27] are devoting to the cases of meromorphic function f or more general difference products. In the following, without special stated, we assume that c is a nonzero constant, n, m, k, s, t are positive integers, $a(z)$ is a nonzero small function with respect to $f(z)$. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a nonzero polynomial, where $a_0, a_1, \ldots, a_n (\neq 0)$ are complex constants and t is the number of the distinct zeros of $P(z)$. Recently, Luo and Lin investigated more generally difference products of entire function and obtained the following result.

Theorem D.[18, Theorem 1] Let f be a transcendental entire function of finite order. For $n > t$, then $P(f)f(z+c) - a(z)$ has infinitely many zeros.

Firstly, we give the following remark to show that the condition $n > t$ in Theorem D is indispensable which is not given in [18].

Remark. If $n = t = 1$, Theorem D is not true, which can be seen by the function $f(z) = e^z + 1$, $e^c = -1$, hence $f(z)f(z+c) - 1 = -e^{2z}$ has no zeros.

If $n = t = 2$, Theorem D also is not true, which can be seen by function $f(z) = \frac{1}{e^z} + 1$, $e^c = -1$, $P(z) = (z + \frac{-1+\sqrt{3}i}{2})(z + \frac{-1-\sqrt{3}i}{2})$, thus, $P(f)f(z+c) - 1 = \frac{-1}{e^{3z}}$ has no zeros.

In fact, for any natural number $n = t$, we can construct an counterexample to show Theorem D is not true by function $f(z) = \frac{1}{e^z} + 1$, $e^c = -1$, $P(z) = (z - 1 - \frac{1}{d_1}) \cdots (z - 1 - \frac{1}{d_{n-1}})$, where $d_i \neq 1, i = 1, 2, \ldots n-1$ are the distinct zero of $z^n - 1 = 0$, thus, we get $P(f)f(z+c) - 1 = \frac{-1}{e^{nz}}$ has no zeros.

As the improvement of Theorem B, it is interesting to investigate the zeros of derivatives of difference polynomials. The present authors [15, Theorem 1.1, Theorem 1.3] have considered the zeros of $[f^n f(z+c)]^{(k)}$ and $[f^n \Delta_c f]^{(k)}$, the results can be stated as follows.

Theorem E. Let f be a transcendental entire function of finite order. If $n \geq k+2$, then $[f(z)^n f(z+c)]^{(k)} - a(z)$ has infinitely many zeros. If $n \geq k+3$, then $[f(z)^n \Delta_c f]^{(k)} - a(z)$ has infinitely many zeros, unless f is a periodic function with period c.

In this paper, we continue to investigate the zeros of derivatives of difference polynomials with more general forms and obtain the following results as the improvements of the Theorem D and Theorem E.

Theorem 1.1. *Let f be a transcendental entire function of $\rho_2(f) < 1$. For $n \geq t(k+1)+1$, then $[P(f)f(z+c)]^{(k)} - a(z)$ has infinitely many zeros.*

Remark. (1). Theorem 1.1 is an improvement of Theorem E of the case $t = 1$ and an improvement of Theorem D of the case $k = 0$.

(2). Theorem 1.1 is not valid for entire function with $\rho_2(f)=1$, which can be seen by $f(z)=e^{e^z}$, $P(z)=z^n$, $k\geq 1$, $e^c=-n$, $a(z)$ is a nonconstant polynomial, thus $[P(f)f(z+c)]^{(k)}-a(z)=-a(z)$ has finitely many zeros.

(3). The condition of $a(z)\neq 0$ can not be removed, which can be seen by function $f(z)=e^z$, $P(z)=z^n$, $e^c=-1$, then $[P(f)f(z+c)]^{(k)}=-(n+1)^k e^{(n+1)z}$ has no zeros.

Theorem 1.2. *Let f be a transcendental entire function of $\rho_2(f)<1$, not a periodic function with period c. If $n\geq (t+1)(k+1)+1$, then $[f(z)^n(\Delta_c f)^s]^{(k)}-a(z)$ has infinitely many zeros.*

Remark. The condition of $a(z)\neq 0$ can not be removed in Theorem 1.2, which can be seen by function $f(z)=e^z$, $P(z)=z^n$ $e^c=2$, then $[P(f)\Delta_c f]^{(k)}=(n+1)^k e^{(n+1)z}$ has no zeros.

For the case of transcendental meromorphic functions of Theorem 1.1 and Theorem 1.2, we obtian the next results.

Theorem 1.3. *Let f be a transcendental meromorphic function of $\rho_2(f)<1$. For $n\geq t(k+1)+5$, then $[P(f)f(z+c)]^{(k)}-a(z)$ has infinitely many zeros.*

Remark. Theorem 1.3 also partly answer the question raised by Luo and Lin [18, P. 448].

Theorem 1.4. *Let f be a transcendental meromorphic function of $\rho_2(f)<1$. For $n\geq (t+2)(k+1)+3+s$, then $[P(f)(\Delta_c f)^s]^{(k)}-a(z)$ has infinitely many zeros.*

Corollary 1.5. *Let $P(z),Q(z),H(z)$ be nonzero polynomials. If $H(z)$ is a nonconstant polynomial, then the nonlinear difference-differential equation*

$$[P(f)f(z+c)]^{(k)}-P(z)=Q(z)e^{H(z)} \tag{1.1}$$

has no transcendental entire (meromorphic) solution of $\rho_2(f)<1$, provided that $n\geq t(k+1)+1$ $(n\geq t(k+1)+5)$. If $H(z)$ is a constant, then (1.1) has no transcendental entire solutions of $\rho_2(f)<1$, and (1.1) has no transcendental meromorphic solutions of $\rho_2(f)<1$ provided that $n\geq 2$.

Corollary 1.6. *Let $P(z),Q(z),H(z)$ be nonzero polynomials. If $H(z)$ is a nonconstant polynomial, then the nonlinear difference-differential equation*

$$[P(f)(\Delta_c f)^s]^{(k)}-P(z)=Q(z)e^{H(z)} \tag{1.2}$$

has no transcendental entire (meromorphic)solution of $\rho_2(f)<1$, provided that $n\geq (t+1)(k+1)+s+1$ $(n\geq (t+2)(k+1)+3+s)$. If $H(z)$ is a constant, then (1.2) has no transcendental entire solutions of $\rho_2(f)<1$, and (1.2) has no transcendental meromorphic solutions of $\rho_2(f)<1$ provided that $n\geq 3$, unless f is a periodic function with period c.

About the uniqueness of difference products of entire functions, some results can be found in [14, 15, 16, 18, 21, 27]. The main purpose is to obtain the relationships between f and g when $P(f)f(z+c)$ and $P(g)g(z+c)$ sharing one common value. In fact, two special types $P(z)=z^n$ and $P(z)=z^n(z^m-1)$ always be considered. Luo and Lin [18, Theorem 2] also considered the general case of $P(z)$. Corresponding to the above theorems of this paper, it is necessary to consider the uniqueness of derivative of difference polynomials sharing one common value. The present

authors [15, Theorem 1.5] have considered the uniqueness about $[f^n f(z+c)]^{(k)}$ and $[g^n g(z+c)]^{(k)}$ sharing one common value, the result can be stated as follows.

Theorem F. Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order, $n \geq 2k+6$. If $[f(z)^n f(z+c)]^{(k)}$ and $[g(z)^n g(z+c)]^{(k)}$ share the value 1 CM, then either $f(z) = c_1 e^{Cz}$, $g(z) = c_2 e^{-Cz}$, where c_1, c_2 and C are constants satisfying $(-1)^k (c_1 c_2)^n [(n+1)C]^{2k} = 1$ or $f = tg$, where $t^{n+1} = 1$.

In this paper, we consider the entire functions of $\rho_2(f) < 1$ and get the following theorems.

Theorem 1.7. *Let $f(z)$ and $g(z)$ be transcendental entire functions of $\rho_2(f) < 1$, $n \geq 2k+m+6$. If $[f^n(f^m-1)f(z+c)]^{(k)}$ and $[g^n(g^m-1)g(z+c)]^{(k)}$ share the value 1 CM, then $f = tg$, where $t^{n+1} = t^m = 1$.*

Theorem 1.8. *The conclusion of Theorem 1.7 is also valid, if $n \geq 5k+4m+12$ and $[f^n(f^m-1)f(z+c)]^{(k)}$ and $[g^n(g^m-1)g(z+c)]^{(k)}$ share the value 1 IM.*

2. Some Lemmas

For a finite order transcendental meromorphic function f, the difference logarithmic derivative lemma, given by Chiang and Feng [4, Corollary 2.5], Halburd and Korhonen [5, Theorem 2.1], [7, Theoem 5.6], plays an important part in considering the difference Nevanlinna theory. Afterwards, R. G. Halburd, R. J. Korhonen and K. Tohge improved the condition of growth from finite order to $\rho_2(f) < 1$ as follows.

Lemma 2.1. [6, Theorem 5.1] *Let f be a transcendental meromorphic function of $\rho_2(f) < 1$, $\varsigma < 1$, ε is a enough small number. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\varsigma-\varepsilon}}\right) = S(r,f), \tag{2.1}$$

for all r outside of a set of finite logarithmic measure.

Lemma 2.2. [6, Lemma 8.3] *Let $T : [0,+\infty) \to [0,+\infty)$ be a non-decreasing continuous function and let $s \in (0,\infty)$. If the hyper order of T is strictly less that one, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \varsigma < 1, \tag{2.2}$$

and $\delta \in (0, 1-\varsigma)$, then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right) \tag{2.3}$$

for all r runs to infinity outside of a set of finite logarithmic measure.

Thus, from Lemma 2.2, we get the following lemma.

Lemma 2.3. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$. Then,*

$$T(r, f(z+c)) = T(r,f) + S(r,f) \tag{2.4}$$

and

$$N(r, f(z+c)) = N(r,f) + S(r,f), \qquad N\left(r, \frac{1}{f(z+c)}\right) = N(r, \frac{1}{f}) + S(r,f). \tag{2.5}$$

Combining the method of proof of [18, Lemma 5] with Lemma 2.1, we can get the following Lemma 2.4–Lemma2.7.

Lemma 2.4. *Let $f(z)$ be a transcendental entire function of $\rho_2(f) < 1$. If $F = P(f)f(z+c)$, then*

$$T(r, F) = T(r, P(f)f(z)) + S(r, f) = (n+1)T(r, f) + S(r, f). \tag{2.6}$$

Lemma 2.5. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$. If $F = P(f)f(z+c)$, then*

$$(n-1)T(r, f) + S(r, f) \le T(r, F) \le (n+1)T(r, f) + S(r, f). \tag{2.7}$$

Proof. Since $F(z) = P(f)f(z+c)$, then

$$\frac{1}{P(f)f} = \frac{1}{F}\frac{f(z+c)}{f(z)}. \tag{2.8}$$

Using the first and second main theorem, Lemma 2.1 and the standard Valrion-Monko's theorem [20], from (2.8), we get

$$\begin{aligned}
(n+1)T(r, f) &\le T(r, F(z)) + T(r, \frac{f(z+c)}{f(z)}) + O(1) \\
&\le T(r, F(z)) + m(r, \frac{f(z+c)}{f(z)}) + N(r, \frac{f(z+c)}{f(z)}) + O(1) \\
&\le T(r, F(z)) + N(r, \frac{f(z+c)}{f(z)}) + S(r, f) \\
&\le T(r, F(z)) + 2T(r, f) + S(r, f), \qquad (2.9)
\end{aligned}$$

hence, we get $T(r, F) \ge (n-1)T(r, f) + S(r, f)$. It is easy to get $T(r, F) \le (n+1)T(r, f) + S(r, f)$. Thus, (2.7) follows. □

Remark. The inequality (2.7) can not be improved by the following two examples. If $f(z) = \tan z$, $P(z) = z^n$, $c_1 = \frac{\pi}{2}$, then

$$T(r, P(z)f(z+c_1)) = -\tan^{n-1} z = (n-1)T(r, f) + S(r, f).$$

If $f(z) = \tan z$, $P(z) = z^n$, $c_2 = \pi$, then

$$T(r, P(z)f(z+c_2)) = \tan^{n+1} z = (n+1)T(r, f) + S(r, f).$$

Lemma 2.6. *Let $f(z)$ be a transcendental entire function of $\rho_2(f) < 1$. Then,*

$$nT(r, f) + S(r, f) \le T(r, P(f)[f(z+c) - f(z)]^s) \le (n+s)T(r, f) + S(r, f). \tag{2.10}$$

Remark. The inequality (2.10) can not be improved by the following two examples. If $f(z) = e^z$, $e^c = 2$, then

$$T(r, f(z)^n[f(z+c) - f(z)]^s) = T(r, e^{(n+s)z}) = (n+s)T(r, f) + S(r, f).$$

If $f(z) = e^z + z$, $c = 2\pi i$, then

$$T(r, f(z)^n[f(z+c) - f(z)^s]) = T(r, (2\pi i)^s[e^z + z]^n) = nT(r, f) + S(r, f).$$

Lemma 2.7. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$. Then,*

(2.11)

$$(n-s)T(r, f) + S(r, f) \le T(r, P(f)[f(z+c) - f(z)]^s) \le (n+2s)T(r, f) + S(r, f).$$

The following lemma is needed for the proof of Theorem 1.7. For the case of $k=0$, $m=1$, f and g are transcendental entire functions of finite order, the proof can be found in [27, The proof of Theorem 6].

Lemma 2.8. *Let f and g be transcendental entire functions of $\rho_2(f)<1$, and c be a nonzero constant. If $n\geq m+5$ and*

$$[f^n(f^m-1)f(z+c)]^{(k)}=[g^n(g^m-1)g(z+c)]^{(k)}, \tag{2.12}$$

then $f=tg$, and $t^{n+1}=t^m=1$.

Proof. From (2.12), we get $f^n(f^m-1)f(z+c)=g^n(g^m-1)g(z+c)+Q(z)$, where $Q(z)$ is a polynomial of degree at most $k-1$. If $Q(z)\not\equiv 0$, then we have

$$\frac{f^n(f^m-1)f(z+c)}{Q(z)}=\frac{g^n(g^m-1)g(z+c)}{Q(z)}+1.$$

From the second main theorem of Nevanlinna and Lemma 2.4, we have

$$\begin{aligned}(n+m+1)T(r,f) &= T(r,\frac{f^n(f^m-1)f(z+c)}{Q(z)})+S(r,f)\\ &\leq \overline{N}(r,\frac{f^n(f^m-1)f(z+c)}{Q(z)})+\overline{N}(r,\frac{Q(z)}{f^n(f^m-1)f(z+c)})\\ &\quad +\overline{N}(r,\frac{Q(z)}{g^n(g^m-1)g(z+c)})+S(r,f)\\ &\leq \overline{N}(r,\frac{1}{f^n(f^m-1)})+\overline{N}(r,\frac{1}{f(z+c)})+\overline{N}(r,\frac{1}{g^n(g^m-1)})\\ &\quad +\overline{N}(r,\frac{1}{g(z+c)})+S(r,f)\\ &\leq (m+2)T(r,f)+(m+2)T(r,g)+S(r,f)+S(r,g). \qquad (2.13)\end{aligned}$$

Similarly as above, we have

$$(n+m+1)T(r,g)\leq (m+2)T(r,f)+(m+2)T(r,g)+S(r,f)+S(r,g).$$

Thus, we get

$$(n+m+1)[T(r,f)+T(r,g)]\leq 2(m+2)[T(r,f)+T(r,g)]+S(r,f)+S(r,g).$$

which is a contradiction with $n\geq m+5$. Hence, we get $P(z)\equiv 0$, which implies that

$$f^n(f^m-1)f(z+c)=g^n(g^m-1)g(z+c). \tag{2.14}$$

Let $G(z)=\frac{f(z)}{g(z)}$. Assume that $G(z)$ is nonconstant. From (2.14), we have

$$g(z)^m=\frac{G(z)^nG(z+c)-1}{G(z)^{n+m}G(z+c)-1}. \tag{2.15}$$

If 1 is a Picard value of $G(z)^{n+m}G(z+c)$, then applying the second main theorem, we get

$$\begin{aligned} T(r,G^{n+m}G(z+c)) &\le \overline{N}(r,G^{n+m}G(z+c))+\overline{N}(r,\frac{1}{G^{n+m}G(z+c)}) \\ &+ \overline{N}(r,\frac{1}{G^{n+m}G(z+c)-1})+S(r,G) \\ &\le 2T(r,G(z))+2T(r,G(z+c))+S(r,G) \\ &\le 4T(r,G(z))+S(r,G). \end{aligned} \tag{2.16}$$

Combining (2.16) with Lemma 2.5, we have $(n+m-1)T(r,G)\le 4T(r,G(z))+S(r,G)$, which is a contradiction with $n\ge m+5$. Therefore, 1 is not a Picard value of $G(z)^{n+m}G(z+c)$. Thus, there exists z_0 such that $G(z_0)^{n+m}G(z_0+c)=1$. The following, we may distinguish two cases.

Case 1. $G(z)^{n+m}G(z+c)\not\equiv 1$. From (2.15) and $g(z)$ is an entire function, then we get $G(z_0)^nG(z_0+c)=1$, thus $G(z_0)^m=1$. Therefore,

$$\overline{N}(r,\frac{1}{G^{n+m}G(z+c)-1})\le \overline{N}(r,\frac{1}{G^m-1})\le mT(r,G)+S(r,G). \tag{2.17}$$

By (2.17) and Lemma 2.3, applying the second main theorem, we get

$$\begin{aligned} T(r,G^{n+m}G(z+c)) &\le \overline{N}(r,G^{n+m}G(z+c))+\overline{N}(r,\frac{1}{G^{n+m}G(z+c)}) \\ &+\overline{N}(r,\frac{1}{G^{n+m}G(z+c)-1})+S(r,G) \\ &\le (m+2)T(r,G(z))+2T(r,G(z+c))+S(r,G) \\ &\le (m+4)T(r,G(z))+S(r,G). \end{aligned} \tag{2.18}$$

On the other hand, we have

$$\begin{aligned} (n+m)T(r,G) &= T(r,G^{n+m}) \\ &\le T(r,G^{n+m}G(z+c))+T(r,G(z+c))+O(1) \\ &\le (m+5)T(r,G(z))+S(r,G), \end{aligned} \tag{2.19}$$

which contradicts $n\ge m+5\ge 6$.

Case 2. $G(z)^{n+m}G(z+c)\equiv 1$, thus,

$$\begin{aligned} (n+m)T(r,G) &= T(r,G(z+c))+S(r,G) \\ &= T(r,G(z))+S(r,G), \end{aligned} \tag{2.20}$$

which also is a contradiction with $n\ge m+5$. Thus, G must be a constant and $f(z)=tg(z)$, where t is a non-zero constant. From $f^n(f^m-1)f(z+c)\equiv g^n(g^m-1)g(z+c)$, we know that $t^m=1$ and $t^{n+1}=1$, n,m are positive integers. $\square$

The following result is related to the growth of solutions of linear difference equation and is needed for the proof of Lemma 2.10, was given by Li and Gao [17, Theorem 2.1]. Here, we give the version with small changes of the type of equation (2.21), the proof are similar.

Lemma 2.9. *Let $a_0(z),a_1(z),\cdots,a_n(z),b(z)$ be polynomials such that $a_0(z)a_n(z)\not\equiv 0$, let c_j be constants and*

$$\deg(\sum_{\deg a_j=d} a_j)=d,$$

where $d = \max_{0\leq j\leq n}\{\deg a_j\}$. *If* $f(z)$ *is a transcendental meromorphic solution of*

$$\sum_{j=0}^{n} a_j(z)f(z+c_j) = b(z), \tag{2.21}$$

then $\rho(f) \geq 1$.

Lemma 2.10. *If* $n \geq k+1$, *then there are no transcendental entire functions* f *and* g *with hyper order less than one, satisfying*

$$[f^n(f^m-1)f(z+c)]^{(k)} \cdot [g^n(g^m-1)g(z+c)]^{(k)} = 1. \tag{2.22}$$

Proof. Assume that f and g satisfy (2.22) and f and g are transcendental entire functions of hyper order less than one. Since $n \geq k+1$, from (2.22), we get f and g have no zeros. Thus, $f(z) = e^{b(z)}$ and $g(z) = e^{d(z)}$, where $b(z), d(z)$ are entire functions with order less than one. Thus, substitute f and g into (2.22), we get

$$[e^{nb(z)}(e^{mb(z)}-1)e^{b(z+c)}]^{(k)}[e^{nd(z)}(e^{md(z)}-1)e^{d(z+c)}]^{(k)} = 1$$

Let $(n+m)b(z)+b(z+c) = B_1(z)$, $nb(z)+b(z+c) = B_2(z)$ and $(n+m)d(z)+d(z+c) = D_1(z)$, $nd(z)+d(z+c) = D_1(z)$.

If $k = 1$, we have

$$[B_1'(z)e^{B_1(z)} - B_2'(z)e^{B_2(z)}][D_1'(z)e^{D_1(z)} - D_2'(z)e^{D_2(z)}] = 1,$$

which implies that $e^{B_2(z)}[B_1'(z)e^{B_1(z)-B_2(z)} - B_2'(z)]$ has no zeros. If $B_1' \neq 0$, remark that 0 is the Picard exceptional value of $e^{B_1(z)-B_2(z)}$, then we get $B_2'(z)$ must be zero, thus B_2 must be a constant. From Lemma 2.9 and $nb(z)+b(z+c) = B_2$, we get $\rho(b(z)) \geq 1$, thus $\rho_2(f) \geq 1$, which is a contradiction. If $B_1' = 0$, then B_1 must be a constant, which also induces that $\rho_2(f) \geq 1$, a contradiction.

If $k = 2$, by calculation, then we have $e^{B_2(z)}[(B_1''(z) + B_1'^2(z))e^{B_1(z)-B_2(z)} - (B_2''(z) + B_2'^2(z))]$ has no zeros. If $B_1'' + B_1'^2 \neq 0$, then $B_2'' + B_2'^2 = 0$. If B_2 is transcendental entire, then we get

$$m(r, B_2') = m(r, \frac{B_2''}{B_2'}) = S(r, B_2'),$$

which is a contradiction with B_2' is transcendental entire. If B_2 is a polynomial, from Lemma 2.9, which also induces that $\rho_2(f) \geq 1$, a contradiction. If $B_1'' + B_1'^2 = 0$, similar as above, we get a contradiction. For any $k \geq 2$, using the similar method as above, we can get the proof of Lemma 2.10. □

Let p be a positive integer and $a \in \mathbb{C}$. We denote by $N_p(r, \frac{1}{f-a})$ the counting function of the zeros of $f-a$ where an m-fold zero is counted m times if $m \leq p$ and p times if $m > p$.

Lemma 2.11. *Let* f *be a nonconstant meromorphic function, and* p, k *be positive integers. Then*

$$T(r, f^{(k)}) \leq T(r,f) + k\overline{N}(r,f) + S(r,f). \tag{2.23}$$

$$N_p(r, \frac{1}{f^{(k)}}) \leq T(r, f^{(k)}) - T(r,f) + N_{p+k}(r, \frac{1}{f}) + S(r,f), \tag{2.24}$$

$$N_p(r, \frac{1}{f^{(k)}}) \leq k\overline{N}(r,f) + N_{p+k}(\frac{1}{f}) + S(r,f), \tag{2.25}$$

Lemma 2.12. [25, Lemma 3] *Let F and G be nonconstant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:*

(i) $\max\{T(r,F),T(r,G)\} \le N_2(r,\frac{1}{F})+N_2(r,F)+N_2(r,\frac{1}{G})+N_2(r,G)+S(r,F)+S(r,G)$,

(ii) $F=G$,

(iii) $F\cdot G=1$.

For the proof of Theorem 1.8, we need the following lemma.

Lemma 2.13. [24, Lemma 2.3] *Let F and G be nonconstant meromorphic functions sharing the value 1 IM. Let*

$$H=\frac{F''}{F'}-2\frac{F'}{F-1}-\frac{G''}{G'}+2\frac{G'}{G-1}.$$

If $H\not\equiv 0$, then

$$\begin{aligned} T(r,F)+T(r,G) \le\ & 2\left(N_2(r,\frac{1}{F})+N_2(r,F)+N_2(r,\frac{1}{G})+N_2(r,G)\right)\\ &+\ 3\left(\overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,G)+\overline{N}(r,\frac{1}{G})\right)\\ &+\ S(r,F)+S(r,G). \end{aligned} \tag{2.26}$$

3. Proofs of Theorem 1.1 and Theorem 1.2

Let $F(z)=P(f)f(z+c)$. From Lemma 2.4, we know that $F(z)$ is not a constant, and $S(r,F)=S(r,F^{(k)})=S(r,f)$ follows. Assume that $F(z)^{(k)}-\alpha(z)$ has only finitely many zeros, combining the second main theorem for three small functions [9, Theorem 2.5] and (2.24) with f is a transcendental entire function, then we get

$$\begin{aligned} T(r,F^{(k)}) \le\ & \overline{N}(r,F^{(k)})+\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{F^{(k)}-\alpha(z)})+S(r,F^{(k)})\\ &\le N_1(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{F^{(k)}-\alpha(z)})+S(r,F^{(k)})\\ &\le T(r,F^{(k)})-T(r,F)+N_{k+1}(r,\frac{1}{F})+S(r,F^{(k)}). \end{aligned} \tag{3.1}$$

Combining (2.7) with (3.1), it implies that

$$\begin{aligned} (n+1)T(r,f)+S(r,f) =\ & T(r,F)\le N_{k+1}(r,\frac{1}{F})+S(r,f)\\ \le\ & t(k+1)\overline{N}(r,\frac{1}{f})+N(r,\frac{1}{f(z+c)})+S(r,f)\\ \le\ & [t(k+1)+1]T(r,f)+S(r,f), \end{aligned} \tag{3.2}$$

which is a contradiction with $n\ge t(k+1)+1$. Thus, Theorem 1.1 is proved. Set $G(z)=P(f)[\Delta_c f]^s$. If $G(z)^{(k)}-\alpha(z)$ has only finitely many zeros, using the similar method as above, from Lemma 2.6, then we get

$$\begin{aligned} nT(r,f)+S(r,f) \le\ & T(r,G)\le N_{k+1}(r,\frac{1}{G})+S(r,f)\\ \le\ & t(k+1)\overline{N}(r,\frac{1}{f})+(k+1)\overline{N}(r,\frac{1}{f(z+c)-f(z)})+S(r,f)\\ \le\ & (t+1)(k+1)T(r,f)+S(r,f), \end{aligned} \tag{3.3}$$

which is a contradiction with $n \geq (t+1)(k+1)+1$. Thus, we get the proof of Theorem 1.2.

4. Proofs of Theorem 1.3 and Theorem 1.4

Let $F(z) = P(f)f(z+c)$. From Lemma 2.5, we know that $F(z)$ is not a constant, and $S(r,F) = S(r,F^{(k)}) = S(r,f)$ follows. Assume that $F(z)^{(k)} - \alpha(z)$ has only finitely many zeros, combining the second main theorem for three small functions [9, Theorem 2.5] and (2.24) with f is a transcendental entire function, then we get

$$\begin{aligned} T(r,F^{(k)}) &\leq \overline{N}(r,F^{(k)}) + \overline{N}(r,\frac{1}{F^{(k)}}) + \overline{N}(r,\frac{1}{F^{(k)}-\alpha(z)}) + S(r,F^{(k)}) \\ &\leq \overline{N}(r,f) + \overline{N}(r,f(z+c)) + N_1(r,\frac{1}{F^{(k)}}) + \overline{N}(r,\frac{1}{F^{(k)}-\alpha(z)}) + S(r,F^{(k)}) \\ &\leq 2T(r,f) + T(r,F^{(k)}) - T(r,F) + N_{k+1}(r,\frac{1}{F}) + S(r,F^{(k)}). \end{aligned} \tag{4.1}$$

Combining (2.7) with (4.1), it implies that

$$\begin{aligned} (n-1)T(r,f) + S(r,f) &\leq T(r,F) \leq 2T(r,f) + N_{k+1}(r,\frac{1}{F}) + S(r,f) \\ &\leq t(k+1)\overline{N}(r,\frac{1}{f}) + N(r,\frac{1}{f(z+c)}) + 2T(r,f) + S(r,f) \\ &\leq [t(k+1)+3]T(r,f) + S(r,f), \end{aligned} \tag{4.2}$$

which is a contradiction with $n \geq t(k+1)+5$. Thus, Theorem 1.3 is proved. Set $G(z) = P(f)[\Delta_c f]^s$. If $G(z)^{(k)} - \alpha(z)$ has only finitely many zeros, using the similar method as above, from Lemma 2.6, then we get

$$\begin{aligned} &(n-s)T(r,f) + S(r,f) \leq T(r,G) \leq 2T(r,f) + N_{k+1}(r,\frac{1}{G}) + S(r,f) \\ &\leq 2T(r,f) + t(k+1)\overline{N}(r,\frac{1}{f}) + (k+1)\overline{N}(r,\frac{1}{f(z+c)-f(z)}) + S(r,f) \\ &\leq [(t+2)(k+1)+2]T(r,f) + S(r,f), \end{aligned} \tag{4.3}$$

which is a contradiction with $n \geq (t+2)(k+1)+3+s$. Thus, we get the proof of Theorem 1.4.

5. Proof of Theorem 1.7

Let $F = [f^n(f^m-1)f(z+c)]^{(k)}$, $G = [g^n(g^m-1)g(z+c)]^{(k)}$. Thus F and G share the value 1 CM. From (2.23) and f is a transcendental entire function, then

$$T(r,F) \leq T(r,f^n(f^m-1)f(z+c)) + S(r,P(f)f(z+c)). \tag{5.1}$$

Combining (5.1) with (2.4), we have $S(r,F) = S(r,f)$. We also have $S(r,G) = S(r,g)$ from the same reason as above. From (2.24), we obtain

$$\begin{aligned} N_2(r,\frac{1}{F}) &= N_2\left(r,\frac{1}{[f^n(f^m-1)f(z+c)]^{(k)}}\right) \\ &\leq T(r,F) - T(r,f^n(f^m-1)f(z+c)) \\ &+ N_{k+2}(r,\frac{1}{f^n(f^m-1)f(z+c)}) + S(r,f). \end{aligned} \tag{5.2}$$

Thus, from Lemma 2.4 and (5.2), we get

$$\begin{aligned}(n+m+1)T(r,f) &= T(r,f^n(f^m-1)f(z+c))+S(r,f)\\ &\le T(r,F)-N_2(r,\frac{1}{F})+N_{k+2}(r,\frac{1}{f^n(f^m-1)f(z+c)})+S(r,f).\end{aligned}\tag{5.3}$$

From (2.25), we obtain

$$\begin{aligned}N_2(r,\frac{1}{F}) &\le N_{k+2}(r,\frac{1}{f^n(f^m-1)f(z+c)})+S(r,f)\\ &\le (k+2)N(r,\frac{1}{f})+N(r,\frac{1}{f^m-1})+N(r,\frac{1}{f(z+c)})+S(r,f)\\ &\le (k+m+3)T(r,f)+S(r,f).\end{aligned}\tag{5.4}$$

Similarly as above, we obtain

$$\begin{aligned}(n+m+1)T(r,g) &\le T(r,G)-N_2(r,\frac{1}{G})\\ &\quad + N_{k+2}(r,\frac{1}{g^n(g^m-1)g(z+c)})+S(r,g).\end{aligned}\tag{5.5}$$

and

$$N_2(r,\frac{1}{G})\le (k+m+3)T(r,g)+S(r,g).\tag{5.6}$$

If the (i) of Lemma 2.12 is satisfied, implies that

$$\max\{T(r,F),T(r,G)\}\le N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+S(r,F)+S(r,G).$$

Thus, combining above with (5.3)–(5.6), we obtain

$$\begin{aligned}(n+m+1)[T(r,f)+T(r,g)] &\le 2N_{k+2}(r,\frac{1}{f^n(f^m-1)f(z+c)})\\ &\quad + 2N_{k+2}(r,\frac{1}{g^n(g^m-1)g(z+c)})+S(r,f)+S(r,g)\\ &\le 2(k+m+3)[T(r,f)+T(r,g)]+S(r,f)+S(r,g),\end{aligned}\tag{5.7}$$

which is a contradiction with $n\ge 2k+m+6$. Hence, $F=G$ or $F\cdot G=1$. From Lemma 2.8 and Lemma 2.10, we get $f=tg$ for $t^m=t^{n+1}=1$. Thus, we get the proof of Theorem 1.7.

6. Proof of Theorem 1.8

Let $F=[f^n(f^m-1)f(z+c)]^{(k)}$, $G=[g^n(g^m-1)g(z+c)]^{(k)}$. We will show that $F=G$ or $F\cdot G=1$ under the conditions of Theorem 1.8. Assume that $H\not\equiv 0$, from (2.26), we get

$$\begin{aligned}T(r,F)+T(r,G) &\le 2\left(N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})\right)+3\left(\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})\right)\\ &\quad + S(r,F)+S(r,G).\end{aligned}\tag{6.1}$$

Combining above with (5.3)–(5.6) and (2.25), we obtain

$$\begin{aligned}(n\,+\,m+1)(T(r,f)+T(r,g)) &\leq T(r,F)+T(r,G)+N_{k+2}(r,\frac{1}{f^n(f^m-1)f(z+c)})\\ &+\,N_{k+2}(r,\frac{1}{g^n(g^m-1)g(z+c)})-N_2(r,\frac{1}{F})-N_2(r,\frac{1}{G})+S(r,f)+S(r,g)\\ &\leq\,2N_{k+2}(r,\frac{1}{f^n(f^m-1)f(z+c)})+2N_{k+2}(r,\frac{1}{g^n(g^m-1)g(z+c)})\\ &+\,3\left(\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})\right)+S(r,f)+S(r,g)\\ &\leq\,(5k+5m+12)[T(r,f)+T(r,g)]+S(r,f)+S(r,g),\end{aligned}$$

which is a contradiction with $n\geq 5k+4m+12$. Thus, we get $H\equiv 0$. The following proof is trivial, the original idea is devoting to Yang and Yi [26]. Here, we give the complete proof. Integrating H twice, we obtain

$$F=\frac{(b+1)G+(a-b-1)}{bG+(a-b)},G=\frac{(a-b-1)-(a-b)F}{Fb-(b+1)} \tag{6.2}$$

which implies that $T(r,F)=T(r,G)+O(1)$. We divide into three cases as follows:

Case 1. $b\neq 0,-1$. If $a-b-1\neq 0$, then by (6.2), we get

$$\overline{N}(r,\frac{1}{F})=\overline{N}\left(r,\frac{1}{G-\frac{a-b-1}{b+1}}\right). \tag{6.3}$$

By the Nevanlinna second main theorem, (2.24) and (2.25), we have

$$\begin{aligned}(n+m+1)T(r,g)\,&\leq\,T(r,G)+N_k(r,\frac{1}{g^n(g^m-1)g(z+c)})-N(r,\frac{1}{G})+S(r,g)\\ &\leq\,N_k(r,\frac{1}{g^n(g^m-1)g(z+c)})+\overline{N}\left(r,\frac{1}{G-\frac{a-b-1}{b+1}}\right)+S(r,g)\\ &\leq\,(k+m+1)T(r,g)+(k+m+2)T(r,f)+S(r,f)+S(r,g).\end{aligned} \tag{6.4}$$

Similarly, we get

$$(n+m+1)T(r,f)\leq(k+m+1)T(r,f)+(k+m+2)T(r,g)+S(r,f)+S(r,g).$$

Thus, from (6.4) and above, then

$$(n+m+1)[T(r,f)+T(r,g)]\leq(2k+2m+3)[T(r,f)+T(r,g)]+S(r,f)+S(r,g),$$

which is a contradiction with $n\geq 5k+4m+12$. Thus, $a-b-1=0$, then

$$F=\frac{(b+1)G}{bG+1}. \tag{6.5}$$

Since F is an entire function and (6.5), then $\overline{N}(r,\frac{1}{G+\frac{1}{b}})=0$. Using the same method as above, we get

$$\begin{aligned}(n+m+1)T(r,g)\,&\leq\,T(r,G)+N_k(r,\frac{1}{g^n(g^m-1)g(z+c)})-N(r,\frac{1}{G})+S(r,g)\\ &\leq\,N_k(r,\frac{1}{g^n(g^m-1)g(z+c)})+\overline{N}\left(r,\frac{1}{G+\frac{1}{b}}\right)+S(r,g)\\ &\leq\,(k+m+1)T(r,g)+S(r,g),\end{aligned} \tag{6.6}$$

which is a contradiction.

Case 2. $b = 0$, $a \neq 1$. From (6.2), we have

$$F = \frac{G + a - 1}{a}. \tag{6.7}$$

Similarly, we also can get a contradiction, Thus, $a = 1$ follows, it implies that $F = G$.

Case 3. $b = -1$, $a \neq -1$. From (6.2), we obtain

$$F = \frac{a}{a + 1 - G}. \tag{6.8}$$

Similarly, we can get a contradiction, $a = -1$ follows. Thus, we get $F \cdot G = 1$. From Lemma 2.8 and Lemma 2.10, we get $f = tg$ for $t^m = t^{n+1} = 1$. Thus, we get the proof of Theorem 1.8.

7. Discussions

In this paper, we investigated the uniqueness of derivative of difference polynomial of entire functions. It is an open question under what conditions Theorem 1.7 holds for meromorphic functions with $\rho_2(f) < 1$. In addition, if $[f^n(f^m - 1)\Delta_c f]^{(k)}$ and $[g^n(g^m - 1)\Delta_c g]^{(k)}$ share one common value, we believe that $f = tg$ for $t^m = t^{n+1} = 1$. Unfortunately, we have not succeed in proving that.

References

[1] W. Bergweiler and A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order, Revista Matemática Iberoamericana.* **11** (1995), 355–373.

[2] H. H. Chen and M. L. Fang , *On the value distribution of $f^n f'$*, Sci. China Ser. A. **38** (1995), 789–798.

[3] H. H. Chen, *Yoshida functions and Picard values of integral functions and their derivatives,* Bull. Austral. Math. Soc. **54** (1996), 373–381.

[4] Y. M. Chiang and S. J. Feng, *On the Nevanlinna characteristic $f(z + \eta)$ and difference equations in the complex plane,* The Ramanujan. J. **16** (2008), 105–129.

[5] R. G. Halburd and R. J. Korhonen, *Difference analogue of the lemma on the logarithmic derivative with application to difference equations,* J. Math. Anal. Appl. **314** (2006), 477–487.

[6] R. G. Halburd, R. J. Korhonen and K. Tohge, *Holomorphic cures with shift-invariant hyperplane preimages*, arXiv: 0903-3236.

[7] R. G. Halburd and R. J. Korhonen, *Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations*, J. Phys. A. **40** (2007), 1–38.

[8] W. K. Hayman, *Picard values of meromorphic functions and their derivatives,* Ann. Math. **70** (1959), 9-42.

[9] W. K. Hayman, *Meromorphic functions.* Oxford at the Clarendon Press, 1964.

[10] I. Laine, *Nevanlinna Theory and Complex Differential Equation.* Studies in Mathematics 15, Walter de Gruyter, Berlin-New (1993).

[11] I. Laine and C. C. Yang, *Value distribution of difference polynomials,* Proc. Japan Acad. Ser. A **83** (2007), 148–151.

[12] K. Liu and L. Z. Yang, *Value distribution of the difference operator*, Arch. Math. **92** (2009), 270–278.

[13] K. Liu, *Value distribution of differences of meromorphic functions,* to appear in Rocky Mountain J. Math.

[14] K. Liu, X. L Liu, and T. B Cao, *Value distributions and uniqueness of difference polynomials,* Advances in Difference Equations Volume (2011), Article ID 234215, pp.12.

[15] K. Liu, X. L Liu, and T. B Cao, *Some results on zeros and uniqueness of difference-differential polynomials.* Submitted.

[16] K. Liu, C. H. Zhang, and L. Z. Yang, *Uniqueness of entire functions and difference polynomials.* Submitted.

[17] S. Li and Z. S. Gao, *Finite order meromorphic solutions of linear difference equations*, Proc. Japan Acad. Ser. A **87** (2011), 73–76.
[18] X. D. Luo and W. C. Lin, *Value sharing results for shifts of meromorphic functions*, J. Math. Anal. Appl, **377** (2011) 441-449
[19] E. Mues, *Über ein Problem von Hayman,* Math. Z, **164** (1979), 239–259.
[20] A. Z. Mohon'ho, *The Nevanlinna characteristics of certain meromorphic functions*, Teor. Funktsii Funktsional. Anal. i Prilozhen. **14** (1971), 83–87 (Russian).
[21] X. G. Qi, L. Z. Yang and K. Liu, *Uniqueness and periodicity of meromorphic functions concerning difference operator*, Comput. Math. Appl **60** (2010), 1739–1746.
[22] Y. F. Wang, *On Mues conjecture and Picard values,* Sci. China. **36** (1993), 28–35.
[23] Y. F. Wang and M. L. Fang, *Picard values and normal families of meromorphic functions with multiple zeros,* Acta Math. Sinica, **14** (1) (1998), 17–26.
[24] J. F. Xu and H. X. Yi, *Uniqueness of entire functions and differential polynomials*, Bull. Korean Math. Soc. **44** (2007), 623–629.
[25] C. C. Yang and X. H. Hua, *Uniqueness and value sharing of meromorphic functions,* Ann. Acad. Sci. Fenn. Math. **22** (1997), 395–406.
[26] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions.* Kluwer Academic Publishers (2003).
[27] J. L. Zhang *Value distribution and shared sets of differences of meromorphic functions*, J. Math. Anal. Appl, **367** (2010), 401–408.

Department of Mathematics, Nanchang University, Nanchang, Jiangxi, 330031, P.R. China
E-mail address: liukai418@126.com

Department of Mathematics, Nanchang University, Nanchang, Jiangxi, 330031, P.R. China
E-mail address: sdliuxinling@hotmail.com

Ting-Bin Cao
Department of Mathematics, Nanchang University, Nanchang, Jiangxi, 330031, P.R. China
E-mail address: tbcao@ncu.edu.cn